数理解析研究所講究録
1117 巻 1999 年 9-27

SEMILINEAR ELLIPTIC EQUATION ON A THIN NETWORK-SHAPED DOMAIN

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§1. INTRODUCTION

We consider the following semilinear elliptic equation in a thin network-shaped domain $\Omega(\zeta)\subset\mathbf{R}^{n}$ $(n\geq 3)$ with variable thickness (see Figure 1):

$$\left\{\begin{array}{ll}\Delta u+f(u)=0 & \text{in } \Omega(\zeta),\\ \dfrac{\partial u}{\partial\nu}=0 & \text{on } \partial\Omega(\zeta)\end{array}\right. \tag{1.1}$$

where ν denotes the unit outward normal vector on $\partial\Omega(\zeta)$ and f is a real valued smooth function on $\mathbf{R}$. We consider a situation that $\Omega(\zeta)$ approaches a certain geometric graph $\mathcal{G}$ when ζ tends to zero (see Figure 2). In this paper, we study the asymptotic behavior of the solutions of (1.1) as $\zeta\rightarrow 0$.

Many researchers have studied partial differential equations on thin domains and associated low dimensional equations. Among them, Yanagida [8] has studied the existence of a stable stationary solution of reaction-diffusion equations on thin tubular domains when an associated one-dimensional equation has a stable stationary solution and in [9], classified geometric graphs according to stability of non-constant steady states of a reaction-diffusion equation. Hale and Raugel [3] have studied the upper semi-continuity at $\zeta=0$ of the attractors of reaction-diffusion equations on a thin L-shaped domain of $\mathbf{R}^{2}$.

In our previous work [11], we specified a network-shaped domain constructed by several self similar regions which approach points and several cylindrical regions which approach straight line segments and we considered the convergence of solutions of (1.1) on that domain when the domain degenerates into the graph. In this paper, we present generalized results than the results of [11] in the sense that thin portions of network-shaped domains are not necessarily cylindrical regions.

An outline of this paper is as follows: In §2, we consider (1.1) on a special network-shaped domain. This domain $\Omega(\zeta)$ approaches a geometric graph such that several smooth arcs meet one point. In this situation, we prove that the solution of (1.1) converges to a solution of an associated limit equation which is a certain system of ordinary differential equations (cf. Theorem 2.1). In §3, we consider a certain inverse

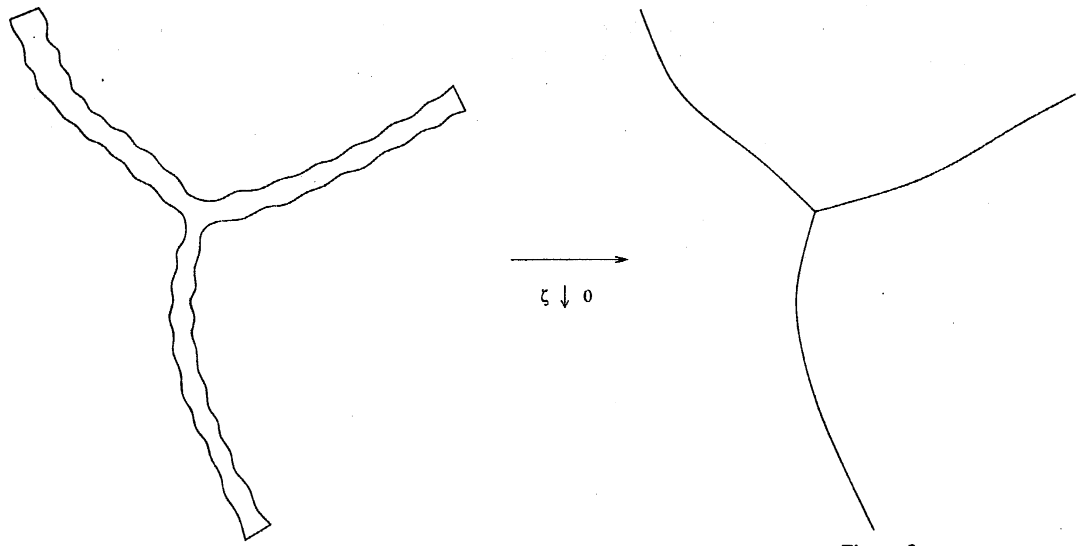

Figure 1 Figure 2

problem of Theorem 2.1, namely, we prove that if the linearized equation around a solution of the limit equation has no zero eigenvalue, then (1.1) has a solution which approaches the solution of the limit equation (cf. Theorem 3.1).

Acknowledgment. I wish to express my sincere gratitude to Professor Shuichi Jimbo for valuable advice and comments.

§2. Simple case

We define a simple network-shaped domain $\Omega(\zeta)$ as follows: We first specify a connected geometric graph $\mathcal{G}$ such that several smooth arcs meet one point, that is, let O be a point of $\mathbf{R}^n$ and p_i a C^∞ mapping from an interval $[0, l_i]$ to $\mathbf{R}^n$ with $p_i(0) = O$ and $|dp_i/ds(s)| = 1$ for $i = 1, \ldots, N$ where s denotes the arc length parameter and l_i is the length of the arc $P_i = \{p_i(s) : 0 < s < l_i\}$. We assume $dp_i/ds(0) \neq dp_{i'}/ds(0)$ $(i \neq i')$ and the graph $\mathcal{G} = \{O\} \cup \bigcup_{i=1}^N P_i$ dose not intersect itself, that is, $\mathcal{G}$ satisfies the following condition: For $x \in \mathcal{G} \setminus \{O\}$ there exists a neighborhood U of x of $\mathbf{R}^n$ such that $U \cap \overline{\mathcal{G}} = U \cap \overline{P_i}$ with $x \in \overline{P_i} \setminus \{O\}$. In this section and §3, we put O the origin to simplify an argument.

Let $Q_i(s)$ be an $(n-1)$-dimensional bounded domain with a smooth boundary which depends on $s \in [0, l_i]$ smoothly, that is, for $t \in [0, l_i]$ and a neighborhood $I \ni t$, there exists a C^3-diffeomorphism $g(s, \cdot) : Q_i(t) \ni \tilde{\xi} \mapsto g(s, \tilde{\xi}) \in Q_i(s)$ for $s \in I$ such that $g(\cdot, \cdot)$ is a C^3-mapping from $I \times Q_i(t)$ to $\mathbf{R}^{n-1}$ with $\|g\|_{C^3(I \times Q_i(t))} < \infty$ and

$$\lim_{s \to t} \|g(s, \tilde{\xi}) - \tilde{\xi}\|_{C^3(Q_i(t))} = 0 \tag{2.1}$$

where $\tilde{\xi} = (\xi_2, \ldots, \xi_n) \in \mathbf{R}^{n-1}$.

For $i = 1, \ldots, N$, let $q_{i,1}(s)$ be $dp_i/ds(s)$ and let $\{q_{i,1}(s), q_{i,2}(s), \ldots, q_{i,n}(s)\}$ be an orthonormal base of $\mathbf{R}^n$ which depends on $s \in [0, l_i]$ smoothly. We define $S_i(s, \zeta)$ by

$$S_i(s,\zeta) = \left\{ x = p_i(s) + \zeta \sum_{j=2}^{n} y_j q_{i,j}(s) \in \mathbf{R}^n : \tilde{y} \in Q_i(s) \right\}$$

where $\zeta > 0$ is a small parameter and $\tilde{y} = (y_2, \ldots, y_n)$. We remark $S_i(s, \zeta)$ is a subset of the normal plane at $p_i(s)$. We define $D_i(\zeta) \subset \mathbf{R}^n$ by

$$D_i(\zeta) = \{x \in S_i(s,\zeta) : \zeta l \le s < l_i\} \quad (0 < \zeta < \zeta^*)$$

where $\zeta^* > 0$ and $l > 0$ are constants such that $D_i(\zeta) \neq \emptyset$, $D_i(\zeta) \cap D_{i'}(\zeta) = \emptyset$ $(i \neq i')$ and that $\sup\{|x - p_i(s)| : x \in S_i(s,\zeta)\}$ is smaller than the radius of curvature at $p_i(s)$ for any $0 < \zeta < \zeta^*$, that is, the mapping $(s, \tilde{y}) \mapsto x$ defined by $x = p_i(s) + \zeta \sum_{j=1}^{n} y_j q_{i,j}(s)$ has a one-to-one correspondence.

Let $J(\zeta)$ be a connected open set which degenerates into the point O as $\zeta \to 0$ satisfying the following conditions (2.2) to (2.4).

(2.2) $\qquad J(\zeta) \cap D_i(\zeta) = \emptyset,\ \partial J(\zeta) \cap \partial D_i(\zeta) = \overline{S_i(\zeta l, \zeta)}$ for $0 < \zeta < \zeta^*$.

(2.3) $\qquad \partial \left(\bigcup_{i=1}^{N} D_i(\zeta) \cup J(\zeta) \right) \setminus \bigcup_{i=1}^{N} S_i(l_i, \zeta)$ is class C^3.

(2.4) There exists a set $J = \lim_{\zeta \to 0} \zeta^{-1} J(\zeta)$ such that J is a connected open set and there exists a C^3-diffeomorphism $G_\zeta : \tilde{J} \ni y \mapsto G_\zeta(y) \in \zeta^{-1}\tilde{J}(\zeta)$ with $\lim_{\zeta \to 0} \|G_\zeta(y) - y\|_{C^3(\tilde{J})} = 0$ where $\zeta^{-1}J(\zeta) = \{\zeta^{-1}x : x \in J(\zeta)\}$, $\tilde{J}$ is a set defined by

$$\tilde{J} = \bigcup_{i=1}^{N} \left\{ \sum_{j=1}^{n} y_j q_{i,j}(0) : \tilde{y} \in Q_i(0),\ l \le y_1 < 2l \right\} \cup J$$

and $\tilde{J}(\zeta)$ is a subset of $\Omega(\zeta)$ defined by

$$\tilde{J}(\zeta) = \bigcup_{i=1}^{N} \left\{ p_i(s) + \zeta \sum_{i=1}^{n} y_j q_{i,j}(s) : \tilde{y} \in Q_i(s),\ l\zeta \le s < 2l\zeta \right\} \cup J(\zeta).$$

Now, we define a simple network shaped domain $\Omega(\zeta)$ by

$$\Omega(\zeta) = \bigcup_{i=1}^{N} D_i(\zeta) \cup J(\zeta).$$

We prepare a certain system of ordinary differential equations used in the main result in this section. Let $a_i(s)$ be $(n-1)$-dimensional volume of $Q_i(s)$, that is, $a_i(s)$ is a smooth function defined by $a_i(s) = \int_{Q_i(s)} d\tilde{y}$. The system of ODEs is

$$
(2.5) \qquad \begin{cases} \dfrac{1}{a_i(s)}\dfrac{d}{ds}\left(a_i(s)\dfrac{d\phi}{ds}(s)\right) + f(\phi(s)) = 0 \text{ on } (0, l_i) & \text{for } i = 1, \dots, N, \\ \phi_1(0) = \cdots = \phi_N(0), \quad \displaystyle\sum_{i=1}^{N} a_i(0)\frac{d\phi_i}{ds}(0) = 0, & \\ \dfrac{d\phi_i}{ds}(l_i) = 0 & \text{for } i = 1, \dots, N \end{cases}
$$

where each ϕ_i is an unknown function on the interval $[0, l_i]$.

We impose the following condition.

$$
(2.6) \qquad f \in C^2(\mathbf{R}), \ \limsup_{\xi \to \infty} f(\xi) < 0, \ \liminf_{\xi \to -\infty} f(\xi) > 0.
$$

Then, the equation (1.1) has at least one solution by the monotone method (see Sattinger [10]). The equation (2.5) is not a usual two-points boundary value problem. However, we can prove the existence of solutions of (2.5) by a manner similar to the monotone method.

Now we present the main result of this section.

Theorem 2.1. *Let $\{\zeta_m\}_{m=1}^{\infty}$ be a positive sequence which satisfies $\lim_{m\to\infty} \zeta_m = 0$ and let $\Omega(\zeta)$ be a simple network shaped domain. Assume that f satisfies* (2.6) *and Ψ_m is any solution of* (1.1) *at $\zeta = \zeta_m$. Then, there exist a solution $\psi = (\psi_1, \dots, \psi_N)$ of* (2.5) *and a subsequence $\{\zeta_{m(k)}\}_{k=1}^{\infty} \subset \{\zeta_m\}_{m=1}^{\infty}$ such that*

$$
\begin{cases} \displaystyle\lim_{k\to\infty} \sup_{x \in J(\zeta_{m(k)})} |\Psi_{m(k)}(x) - \psi_i(0)| = 0 & \text{for } 1 \le i \le N, \\ \displaystyle\lim_{k\to\infty} \sup_{x \in D_i(\zeta_{m(k)})} |\Psi_{m(k)}(x) - \psi_i(s)| = 0 & \text{for } 1 \le i \le N \end{cases}
$$

where $s \in (l\zeta, l_i)$ defined by $S_i(s, \zeta) \ni x$ for $x \in D_i(\zeta)$.

Proof of Theorem 2.1. Let M_1 be a constant $M_1 = \max\{|\xi| : f(\xi) = 0\}$. Then, we have

$$
(2.7) \qquad \sup_{x \in \Omega(\zeta)} |\Psi_m(x)| \le M_1
$$

by the maximum principle. Let $\delta > 0$ be a small constant and we take finite constants $s_{i,j} \in (0, l_i)$ $(1 \le i \le N,\ 1 \le j \le N(i))$ such that $s_{i,1} < \delta/2$, $l_i - s_{i,N(i)} < \delta/2$ and that $0 < s_{i,j+1} - s_{i,j} < \delta/2$ and we put $s_{i,0} = \zeta l$ and $s_{i,N(i)+1} = l_i$. We define

$D_{i,j}(\zeta) \subset D_i(\zeta)$ as $D_{i,j}(\zeta) = \{x \in S_i(s,\zeta) : s_{i,j-1} < s < s_{i,j+1}\}$ for $1 \le j \le N(i)$. Let $\lambda_1(D_{i,j}(\zeta))$ be the first eigenvalue of the Laplacian operator with a certain boundary condition, that is,

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } D_{i,j}(\zeta), \\ u = 0 & \text{on } T, \\ \partial u / \partial \nu = 0 & \text{on } \partial D_{i,j}(\zeta) \setminus T \end{cases}$$

where $T = \overline{S_i(s_{i,j-1},\zeta) \cup S_i(s_{i,j+1},\zeta)}$ in the case $1 \le j \le N(i)-1$ and $T = \overline{S_i(s_{i,j-1},\zeta)}$ in the case $j = N(i)$. It is well known that $\lambda_1(D_{i,j}(\zeta)) > 0$ and $\lambda_1(D_{i,j}(\zeta)) \to \infty$ as the radius of $D_{i,j}(\zeta)$ goes to zero. Without loss of generality, we may take small constants $\zeta^* > 0$ and $\delta > 0$ satisfying the following conditions (2.8) and (2.9):

$$\begin{aligned} &\min\{\lambda_1(D_{i,j}(\zeta)) : 1 \le i \le N, 1 \le j \le N(i)\} \\ &\quad > \max\{|f'(\xi)| : |\xi| \le M_1 + 1\} \qquad \text{for } \zeta \in (0, \zeta^*] \end{aligned} \tag{2.8}$$

$$\delta < \frac{a_*}{a^*} \min \left\{ \left(\sup_{|\xi| \le 3M_1+1} |f(\xi)| + 1 \right)^{-1/2}, \left(2 \sup_{|\xi| \le 1} |f'(\xi)| \right)^{-1/2} \right\} \tag{2.9}$$

where $a^* = \min\{a_i(s) : 0 \le s \le l_i, 1 \le i \le N\}$ and $a^* = \max\{a_i(s) : 0 \le s \le l_i, 1 \le i \le N\}$.

To see the behavior of Ψ_m on $J(\zeta_m)$, we define $U_m(y)$ as

$$U_m(y) = \Psi_m(x), \ x = \zeta_m G_{\zeta_m}(y), \ (y \in \tilde{J}).$$

Then, we have the following:

Lemma 2.2. *There exist positive constants M_2 and M_3 such that the function U_m restricted on J satisfies $\|U_m\|_{C^2(J)} \le M_2$ and for small ζ_m*

$$\int_J |\nabla_y U_m(y)|^2 dy \le M_3 \zeta_m.$$

Proof of Lemma 2.2. From the definition of $G_\zeta = G_\zeta(y) = (G_{\zeta,1}(y), \ldots, G_{\zeta,n}(y))$, we obtain the Jacobian matrix DG_ζ satisfies

$$DG_\zeta = \left(\frac{\partial G_{\zeta,i}}{\partial y_j} \right)_{ij} = E + o(1) \text{ in } C^2(\tilde{J}) \text{ as } \zeta \to 0$$

where E denotes the identity matrix on $\mathbf{R}$, that is, $\lim_{\zeta \to 0} \|\partial G_{\zeta,i}/\partial y_i - 1\|_{C^2(\tilde{J})} = 0$ and $\lim_{\zeta \to 0} \|\partial G_{\zeta,i}/\partial y_j\|_{C^2(\tilde{J})} = 0 \ (i \ne j)$.

From a simple calculation, U_m satisfies $\mathcal{L}_\zeta U_m(y) + {\zeta_m}^2 f(U_m(y)) = 0$ in $\tilde{J}$ where $\mathcal{L}_\zeta$ is an elliptic differential operator

$$\mathcal{L}_\zeta = \sum_{1 \le i,j \le n} \alpha_{ij}(\zeta, y) \frac{\partial^2}{\partial y_i \partial y_j} + \sum_{1 \le j \le n} \beta_j(\zeta, y) \frac{\partial}{\partial y_j}.$$

Here, the matrix (α_{ij}) satisfies $(\alpha_{ij}) = D{G_\zeta}^{-1} \cdot {}^tD{G_\zeta}^{-1} = E + o(1)$ in $C^2(\tilde{J})$ as $\zeta \to 0$ and β_j $(1 \le j \le n)$ satisfies $\beta_j = o(\zeta)$ in $C^1(\tilde{J})$ as $\zeta \to 0$. We put $T = \partial \tilde{J} \setminus \bigcup_{i=1}^N \overline{\{\sum_{j=1}^n y_j q_{i,j}(0) : y_1 = 2l,\ \tilde{y} \in Q_i(0)\}}$. Then, we obtain $\nu(\zeta G_\zeta(y)) \cdot {}^tD{G_\zeta}^{-1} \cdot {}^t\nabla_y U_m(y) = 0$ on T. Let $\tilde{\nu}(y)$ be the outward normal vector at $y \in T$. We obtain $|\nu(\zeta G_\zeta(y)) \cdot {}^tD{G_\zeta}^{-1} \cdot \tilde{\nu}(y)| = 1 + o(1)$ in $C^0(T)$ as $\zeta \to 0$ and $\|\nu(\zeta G_\zeta(y)) \cdot {}^tD{G_\zeta}^{-1}\|_{C^2(T)} <$ constant for any ζ. Therefore, by (2.7) and applying the Schauder interior estimates and boundary estimates, $\|U_m\|_{C^2(J)}$ is bounded independently of ζ_m.

Changing of variables, we obtain

$$\begin{aligned}
&\int_J \nabla_y U_m(y) \cdot DG_\zeta^{-1} \cdot {}^tDG_\zeta^{-1} \cdot {}^t\nabla_y U_m(y) \det DG_\zeta dy \\
&= \zeta^{2-n} \int_{J(\zeta)} |\nabla \Psi_m|^2 dx \le \zeta^{2-n} \int_{\Omega(\zeta)} |\nabla \Psi_m|^2 dx \\
&= \zeta^{2-n} \int_{\Omega(\zeta)} f(\Psi_m) \Psi_m dx \le \zeta^{2-n} |\Omega(\zeta)| \sup_{|\xi| < M_1} |f(\xi)| M_1
\end{aligned}$$

On the other hand, when $\zeta_m > 0$ is small,

$$\int_J \nabla_y U_m(y) \cdot DG_\zeta^{-1} \cdot {}^tDG_\zeta^{-1} \cdot {}^t\nabla_y U_m(y) \det DG_\zeta dy \ge \frac{1}{2} \int_J |\nabla U_m(y)|^2 dy.$$

Therefore, we complete the proof of Lemma 2.2. $\square$

For $i = 1, \ldots, N$ and for $j = 1, \ldots, N(i)$, to see the behavior of Ψ_m on the $S_i(s_{i,j}, \zeta)$, we define a function $V_m^{i,j}(z)$ $(z \in [-2,2] \times Q_i(s_{i,j}))$ as

$$\begin{aligned}
&V_m^{i,j}(z) = \Psi_m(x), \quad x = p_i(s_{i,j} + \zeta y_1) + \zeta \sum_{k=2}^n y_k q_{i,k}(s_{i,j} + \zeta y_1), \\
&y = (z_1, g(s_{i,j} + \zeta z_1, \tilde{z})), \quad z = (z_1, \tilde{z}) \in [-2,2] \times Q_i(s_{i,j})
\end{aligned}$$

where $\zeta = \zeta_m$ and C^3-diffeomorphism $g(s, \cdot) : Q_i(s_{i,j}) \to Q_i(s)$ satisfies (2.1). Then, we have the following:

Lemma 2.3. *There exist positive constants M_4 and M_5 such that the function $V_m^{i,j}$ restricted on $[-1,1]\times Q_i(s_{i,j})$ satisfies $\|V_m^{i,j}\|_{C^2([-1,1]\times Q)} \le M_4$ and for small ζ_m*

$$\int_{[-1,1]\times Q} |\nabla_z V_m^{i,j}(z)|^2 dz \le M_5 \zeta_m$$

where $Q = Q_i(s_{i,j})$.

Proof of Lemma 2.3. In this proof, we put $t = s_{i,j}$, $V_m = V_m^{i,j}$, $Q = Q_i(s_{i,j})$, $p(s) = p_i(s)$ and $q_j(s) = q_{i,j}(s)$ for short. We remark $p'(s) = q_1(s)$. The Jacobian matrixes satisfy

$$\frac{Dx}{Dy} = \zeta \left({}^{\mathrm{t}}q_1 + \zeta \sum_{j=2}^{n} y_j {}^{\mathrm{t}}q_j{}', {}^{\mathrm{t}}q_2, \dots, {}^{\mathrm{t}}q_n \right),$$

$$\frac{Dy}{Dz} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \zeta\dfrac{\partial g_2}{\partial s} & \dfrac{\partial g_2}{\partial \xi_2} & \dots & \dfrac{\partial g_2}{\partial \xi_n} \\ \vdots & \vdots & & \vdots \\ \zeta\dfrac{\partial g_n}{\partial s} & \dfrac{\partial g_n}{\partial \xi_2} & \dots & \dfrac{\partial g_n}{\partial \xi_n} \end{pmatrix} = E + o(1) \text{ in } C^2 \text{ as } \zeta \to 0$$

where $q_j = q_j(t+\zeta y_1)$, $q_j{}' = q_j{}'(t+\zeta y_1)$, $g = (g_2, \dots, g_n)$, $\partial g_i/\partial s = \partial g_i/\partial s(t+\zeta z_1, \tilde{z})$ and $\partial g_i/\partial \xi_j = \partial g_i/\partial \xi_j(t+\zeta z_1, \tilde{z})$. Then, we have

$$\frac{Dx}{Dy}^{-1} = \zeta^{-1} \begin{pmatrix} q_1 - \dfrac{\zeta\gamma_1}{1+\zeta\gamma_1} q_1 \\ \vdots \\ q_n - \dfrac{\zeta\gamma_n}{1+\zeta\gamma_1} q_1 \end{pmatrix}, \quad \frac{Dy}{Dz}^{-1} = E + o(1) \text{ in } C^2 \text{ as } \zeta \to 0$$

where

$$\gamma_k = \gamma_k(\zeta, y) = \sum_{j=2}^{n} y_j q_j{}'(t+\zeta y_1) \cdot {}^{\mathrm{t}}q_k(t+\zeta y_1).$$

From a simple calculation, V_m satisfies $\mathcal{L}_{\zeta_m} V_m + {\zeta_m}^2 f(V_m) = 0$ in $[-2,2]\times Q$ where $\mathcal{L}_\zeta$ is an elliptic differential operator

$$\mathcal{L}_\zeta = \sum_{1\le i,j\le n} \alpha_{i\,j}(\zeta, z) \frac{\partial^2}{\partial z_i \partial z_j} + \sum_{1\le j\le n} \beta_j(\zeta, z) \frac{\partial}{\partial z_j}.$$

Here, the matrix $(\alpha_{i\,j})_{1\le i,j\le n}$ satisfies

$$\begin{aligned} (\alpha_{i\,j}) &= \zeta^2 \frac{Dy}{Dz}^{-1} \cdot \frac{Dx}{Dy}^{-1} \cdot {}^{\mathrm{t}}\frac{Dx}{Dy}^{-1} \cdot {}^{\mathrm{t}}\frac{Dy}{Dz}^{-1} \\ &= E + o(1) \text{ in } C^2([-2,2]\times Q) \text{ as } \zeta \to 0 \end{aligned}$$

and $\beta_j(\zeta, z) = o(\zeta)$ in $C^1([-2,2]\times Q)$ as $\zeta \to 0$.

We set $T = (-2,2)\times \partial Q$. Then, we obtain

$$\zeta_m \nu(x)\cdot {}^{t}\frac{Dx}{Dy}^{-1}\cdot {}^{t}\frac{Dy}{Dz}^{-1}\cdot {}^{t}\nabla_z V_m(z) = 0 \text{ on } T.$$

Let $\tilde{\nu}(\tilde{z}) = (\tilde{\nu}_2(\tilde{z}),\ldots,\tilde{\nu}_n(\tilde{z}))$ be the outward normal vector at $\tilde{z}\in\partial Q$. Then, $(0,\tilde{\nu}(\tilde{z}))$ is the outward normal vector at $z=(z_1,\tilde{z})\in T$. From the definition of x for $z\in T$, we have $\nu(x)\to\sum_{j=2}^{n}\tilde{\nu}_j(\tilde{z})q_j(t)$ as $\zeta\to 0$, thus we obtain

$$\zeta_m \nu(x)\cdot {}^{t}\frac{Dx}{Dy}^{-1}\cdot {}^{t}\frac{Dy}{Dz}^{-1}\cdot {}^{t}(0,\tilde{\nu}(\tilde{z})) = 1 + o(1) \text{ in } C^0(T) \text{ as } \zeta\to 0,$$

$$\left\|\zeta_m \nu(x)\cdot {}^{t}\frac{Dx}{Dy}^{-1}\cdot {}^{t}\frac{Dy}{Dz}^{-1}\right\|_{C^2(T)} < \text{constant}.$$

Therefore, applying the Schauder estimates, there exists a constant $M_4>0$ such that $\|V_m\|_{C^2([-1,1]\times Q)}\le M_4$.

Changing of variables, we have

$$\begin{aligned}
&\int_{[-1,1]\times Q}\nabla_z V_m(z)\cdot \frac{Dx}{Dz}^{-1}\cdot {}^{t}\frac{Dx}{Dz}^{-1}\cdot {}^{t}\nabla_z V_m(z)\det\frac{Dx}{Dz}dz\\
&=\int_{D(\zeta_m)}|\nabla_x\Psi_m(x)|^2dx \le \int_{\Omega(\zeta_m)}|\nabla_x\Psi_m(x)|^2dx\\
&=\int_{\Omega(\zeta_m)}f(\Psi_m(x))\Psi_m(x)dx \le |\Omega(\zeta_m)|\sup_{|\xi|<M_1}|f(\xi)|M_1
\end{aligned}$$

where $\dfrac{Dx}{Dz}=\dfrac{Dx}{Dy}\cdot\dfrac{Dy}{Dz}$ and $D(\zeta)=\{x\in S_i(s,\zeta): |t-s|<\zeta\}$. On the other hand, for small ζ_m we have

$$\begin{aligned}
&\int_{[-1,1]\times Q}\nabla_z V_m(z)\cdot \frac{Dx}{Dz}^{-1}\cdot {}^{t}\frac{Dx}{Dz}^{-1}\cdot {}^{t}\nabla_z V_m(z)\det\frac{Dx}{Dz}dz\\
&\ge \frac{\zeta_m{}^{n-2}}{2}\int_{[-1,1]\times Q}|\nabla_z V_m(z)|^2dz.
\end{aligned}$$

Thus, we have

$$\int_{[-1,1]\times Q}|\nabla_z V_m(z)|^2dz \le 2\frac{|\Omega(\zeta)|}{\zeta^{n-2}}\sup_{|\xi|<M_1}|f(\xi)|M_1.$$

Therefore, we complete the proof of Lemma 2.3. □

From Lemma 2.2 and Lemma 2.3, applying the Ascoli-Arzelà theorem, there exist a subsequence $\{\zeta_{m(k)}\}_{k=1}^{\infty}\subset\{\zeta_m\}_{m=1}^{\infty}$ and constant functions U_∞ on $\tilde{J}$ and $V_\infty^{i,j}$ on $[-1,1]\times Q_i(s_{i,j})$ $(1\le i\le N,\ 1\le j\le N(i))$ such that $U_{m(k)}\to U_\infty$ in $C^1(\tilde{J})$ and $V_{m(k)}^{i,j}\to V_\infty^{i,j}$ in $C^1([-1,1]\times Q_i(s_{i,j}))$ as $k\to\infty$. From the definition of U_m and $V_m^{i,j}$, we obtain the following:

Lemma 2.4. *There exist a subsequence $\{\zeta_{m(k)}\}_{k=1}^{\infty} \subset \{\zeta_m\}_{m=1}^{\infty}$ and constants ϕ_0 and $\phi_{i,j}$ $(1 \le i \le N,\ 1 \le j \le N(i))$ such that*

$$\lim_{k\to\infty} \sup_{x\in J(\zeta_{m(k)})} |\Psi_{m(k)}(x) - \phi_0| = 0,$$
$$\lim_{k\to\infty} \sup_{x\in S_i(s_{i,j},\zeta_{m(k)})} |\Psi_{m(k)}(x) - \phi_{i,j}| = 0.$$

Hereafter, we denote by same notation $\{\zeta_m\}_{m=1}^{\infty}$ the subsequence $\{\zeta_{m(k)}\}_{k=1}^{\infty}$ for short. To construct an upper solution of Ψ_m on the portion $D_{i,j}(\zeta_m) \subset D_i(\zeta_m)$, we consider the following one-dimensional differential equations on the interval $(s_{i,j-1}, s_{i,j+1})$:

$$(2.10)\qquad \begin{cases} \dfrac{1}{a_i(s)}\dfrac{d}{ds}\left(a_i(s)\dfrac{d\psi}{ds}\right) + f(\psi) + {\zeta_m}^{1/3} = 0 \quad (s_{i,j-1} < s < s_{i,j+1}) \\ \psi(s_{i,j-1}) = \phi_{i,j-1} + \displaystyle\sup_{x\in S_i(s_{i,j-1},\zeta_m)} |\Psi_m(x) - \phi_{i,j-1}|, \\ \psi(s_{i,j+1}) = \phi_{i,j+1} + \displaystyle\sup_{x\in S_i(s_{i,j+1},\zeta_m)} |\Psi_m(x) - \phi_{i,j+1}| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in the case } 1 \le j \le N(i)-1, \\ \dfrac{d\psi}{ds}(s_{i,j+1}) = \zeta_m \qquad\qquad\qquad\quad \text{in the case } j = N(i). \end{cases}$$

Here, we put $\phi_{i,0} = \phi_0$ for convenience. Then, we have the following:

Lemma 2.5. *Let $\delta > 0$ satisfy* (2.9)*. Then, for $i = 1,\dots,N$, $j = 1,\dots,N(i)$ and for any $\zeta_m \le 1$ the equation* (2.10) *has a unique solution $\theta^{u}_{i,j,m}(s)$ $(s_{i,j-1} \le s \le s_{i,j+1})$.*

Proof of Lemma 2.5. In this proof, we put $\zeta = \zeta_m$, $s' = s_{i,j-1}$, $s'' = s_{i,j+1}$, $a(s) = a_i(s)$, $A(s) = \int_{s'}^{s} a_i(t)^{-1}dt$, $b' = \phi_{i,j-1} + \sup\{|\Psi_m(x) - \phi_{i,j-1}| : x \in S_i(s_{i,j-1},\zeta_m)\}$ and $b'' = \phi_{i,j+1} + \sup\{|\Psi_m(x) - \phi_{i,j+1}| : x \in S_i(s_{i,j+1},\zeta_m)\}$ for short. It is easy to see that $s'' - s' < \delta$, $|b'| \le M_1$ and $|b''| \le M_1$ for any ζ. In the case $1 \le j \le N(i)-1$, we put $w(s) = \{b'(A(s'') - A(s)) + b''A(s)\}/A(s'')$. Then, we have $w(s') = b'$, $w(s'') = b''$, $|w(s)| \le M_1$ and $\dfrac{1}{a(s)}\dfrac{d}{ds}\left(a(s)\dfrac{dw}{ds}(s)\right) = 0 \quad (s' < s < s'')$. We define the mapping $\mathcal{F}$ on $C^0([s',s''])$ by

$$\mathcal{F}(\psi)(s) = \int_{s'}^{s} \frac{(A(s'') - A(s))A(t)}{A(s'')}\left(f(\psi(t) + w(t)) + \zeta^{1/3}\right)a(t)dt$$
$$+ \int_{s}^{s''} \frac{A(s)(A(s'') - A(t))}{A(s'')}\left(f(\psi(t) + w(t)) + \zeta^{1/3}\right)a(t)dt.$$

Then, $\mathcal{F}$ is a contraction mapping on $\{\psi \in C^0([s',s'']) : \|\psi\|_{C^0} \le 1\}$ by (2.9) and $\phi = \mathcal{F}(\psi)$ satisfies $\phi(s') = 0$, $\phi(s'') = 0$ and

$$\frac{1}{a(s)}\frac{d}{ds}\left(a(s)\frac{d\phi}{ds}(s)\right) + f(\psi(s) + w(s)) + \zeta^{1/3} = 0 \quad (s' < s < s'').$$

From the contraction mapping theorem, the equation (2.10) has a unique solution.

In the case $j = N(i)$, we put $w(s) = a(s'')\zeta A(s) + b'$ and

$$\mathcal{F}(\psi)(s) = \int_{s'}^{s} A(t)\left(f(\psi(t) + w(t)) + \zeta^{1/3}\right) a(t)dt$$
$$+ \int_{s}^{s''} A(s)\left(f(\psi(t) + w(t)) + \zeta^{1/3}\right) a(t)dt.$$

Then, the equation (2.10) has a unique solution by an argument similar to that of the above cases.

Therefore, we complete the proof of Lemma 2.5. □

We define $b_1^i = b_1^i(x)$, $b_2^i = b_2^i(x) \in \mathbf{R}$ for $x \in \partial D_i(\zeta) \setminus \overline{S_i(\zeta l, \zeta) \cup S_i(l_i, \zeta)}$ as follows: Let $(s, \tilde{y})$ satisfy $x = p_i(s) + \zeta \sum_{j=1}^{n} y_j q_{i,j}(s)$. Let $\kappa^j(x)$ ($j = 1, \ldots, n-2$) be tangent vectors at x on $\partial D_i(\zeta)$ in the normal plane at $p_i(s)$ satisfying that $\kappa^j(x)$ ($1 \le j \le n-2$) are orthogonal to each other. Let $\tilde{\nu} = (\tilde{\nu}_2(s, \tilde{y}), \ldots, \tilde{\nu}_n(s, \tilde{y}))$ be the unit outword normal vector of $\partial Q_i(s)$ at $\tilde{y}$ and we put $\nu_S(x) = \sum_{j=2}^{n} \tilde{\nu}_j(s, \tilde{y}) q_{i,j}(s)$. Then, $q_{i,1}(s)$, $\kappa^j(s)$ ($1 \le j \le n-2$) and $\nu_S(x)$ are orthogonal to each other. Let $x(t)$ be the point of $\partial D_i(\zeta) \cap \overline{S_i(t, \zeta)}$ such that $x(t) - x$ is orthogonal to $\kappa^j(x)$ ($1 \le j \le n-2$) and we define $\kappa(x)$ as

$$\kappa(x) = \lim_{t \to s} \frac{x(t) - x}{t - s}. \tag{2.11}$$

We put $b_1^i(x) = \kappa(x) \cdot {}^t q_{i,1}(s)$ and $b_2^i(x) = \kappa(x) \cdot {}^t \nu_S(x)$. Clearly, we have

$$\begin{aligned} &\kappa(x) = b_1^i(x) q_{i,1}(s) + b_2^i(x) \nu_S(x), \\ &b_1^i(x) = 1 + O(\zeta), \quad b_2^i(x) = O(\zeta) \quad \text{as } \zeta \to 0. \end{aligned} \tag{2.12}$$

Thus, we have

$$\nu(x) = -\frac{b_2^i(x)}{\sqrt{b_1^i(x)^2 + b_2^i(x)^2}} q_{i,1}(s) + \frac{b_1^i(x)}{\sqrt{b_1^i(x)^2 + b_2^i(x)^2}} \nu_S(x). \tag{2.13}$$

Indeed, we put $\tilde{y}(t) = (y_2(t), \ldots, y_n(t)) \in \partial Q_i(t)$ satisfying $x - x(t)$ orthogonal to $\kappa^j(x)$ ($1 \le j \le n-2$) where $x(t) = p_i(t) + \zeta \sum_{j=2}^{n} y_j(t) q_{i,j}(t)$. Then, we have

$$b_1^i(x) = 1 + \zeta \sum_{j=2}^{n} y_j(s) q_{i,j}{}'(s) \cdot {}^t q_{i,1}(s)$$

$$b_2^i(x) = \zeta \sum_{j=2}^{n} \left(y_j{}'(s) \tilde{\nu}_j(s, \tilde{y}(s)) + y_j(s) q_{i,j}{}'(s) \cdot {}^t \nu_S(x)\right)$$

Therefore, we obtain (2.12).

From Lemma 3.1 of Yanagida [8], we obtain

$$\zeta^{n-1}\frac{da_i}{ds}(s) = \int_{\partial S_i(s,\zeta)} b_2^i(x)d\sigma_x \tag{2.14}$$

where $\partial S_i(s,\zeta) = \partial D_i(\zeta) \cap \overline{S_i(s,\zeta)}$.

For $i = 1,\dots,N$ and $j = 1,\dots,N(i)$, we take a fixed point $\tilde{y}^1 \in Q_i(s_{i,j})$ and let $g(s,\cdot) : Q_i(s_{i,j}) \to Q_i(s)$ be C^3-diffeomorphism. We define a function $W^{\mathrm{u}}_{i,j,m}(s,\cdot) = W^{\mathrm{u}}_{i,j,m}(s,\tilde{y})$ on $Q_i(s)$ ($s \in [s_{i,j-1}, s_{i,j+1}]$) by the solution of

$$\begin{cases} \Delta_{\tilde{y}} W = \dfrac{a_i{}'(s)}{a_i(s)}\theta^{\mathrm{u}}_{i,j,m}(s) + \zeta_m{}^{2/3}\int_{\partial Q_i(s)} d\omega_{\tilde{\xi}} & \text{in } Q_i(s) \\ \dfrac{\partial W}{\partial \tilde{\nu}} = \dfrac{b_2^i(x)}{\zeta_m}\theta^{\mathrm{u}}_{i,j,m}(s) + \zeta_m{}^{2/3}a_i(s) & \text{on } \partial Q_i(s) \end{cases} \tag{2.15}$$

satisfying $W(g(s,\tilde{y}^1)) = 1$. To show that $W^{\mathrm{u}}_{i,j,m}$ exists, it is sufficient to show

$$\begin{aligned} &\int_{Q_i(s)} \left\{ \frac{a_i{}'(s)}{a_i(s)}\theta^{\mathrm{u}}_{i,j,m}(s) + \zeta_m{}^{2/3}\int_{\partial Q_i(s)} d\omega_{\tilde{\xi}} \right\} d\tilde{y} \\ &\qquad = \int_{\partial Q_i(s)} \left\{ \frac{b_2^i(x)}{\zeta_m}\theta^{\mathrm{u}}_{i,j,m}(s) + \zeta_m{}^{2/3}a_i(s) \right\} d\omega_{\tilde{y}}. \end{aligned} \tag{2.16}$$

From the definition of a_i, we have

$$\int_{Q_i(s)} \frac{a_i{}'(s)}{a_i(s)}\ \theta^{\mathrm{u}}_{i,j,m}(s)\ d\tilde{y} = a_i{}'(s)\ \theta^{\mathrm{u}}_{i,j,m}(s)$$

From (2.14), we have

$$\begin{aligned} \int_{\partial Q_i(s)} \frac{b_2^i(x)}{\zeta_m}\theta^{\mathrm{u}}_{i,j,m}(s)d\omega_{\tilde{y}} &= \zeta_m{}^{1-n}\int_{\partial S_i(s,\zeta_m)} b_2^i(x)\ \theta^{\mathrm{u}}_{i,j,m}(s)\ d\sigma_x \\ &= a_i{}'(s)\ \theta^{\mathrm{u}}_{i,j,m}(s) \end{aligned}$$

Clearly, we have

$$\int_{Q_i(s)} \zeta_m{}^{2/3}\int_{\partial Q_i(s)} d\omega_{\tilde{\xi}}\ d\tilde{y} = \int_{\partial Q_i(s)} \zeta_m{}^{2/3}a_i(s)\ d\omega_{\tilde{y}}$$

Therefore, we obtain (2.16).

Since $Q_i(s)$ and $g(s,\cdot)$ depend on s smoothly, $W^{\mathrm{u}}_{i,j,m}(s,\tilde{y})$ is a smooth function of $(s,\tilde{y})$. From (2.12), we remark $\|W^{\mathrm{u}}_{i,j,m}\|_{C^2}$ is bounded independently of ζ_m.

For $i = 1,\dots,N$, $j = 1,\dots,N(i)$ and ζ_m, we define a function $\Theta^{\mathrm{u}}_{i,j,m}$ on $D_{i,j}(\zeta_m)$ by

$$\Theta^{\mathrm{u}}_{i,j,m}(x) = \theta^{\mathrm{u}}_{i,j,m}(y_1) + \zeta_m{}^2 W^{\mathrm{u}}_{i,j,m}(y_1,\tilde{y}) + \zeta_m \quad x \in D_{i,j}(\zeta_m)$$

where $y = (y_1,\tilde{y})$ satisfies $x = p_i(y_1) + \zeta_m \sum_{j=1}^n y_j\ q_{i,j}(y_1)$. Then, we have the following:

Lemma 2.6. *The function $\Theta^{\mathrm{u}}_{i,j,m}(x)$ is an upper solution of Ψ_m restricted on $D_{i,j}(\zeta_m)$, that is,*

$$\Psi_m(x) \le \Theta^{\mathrm{u}}_{i,j,m}(x) \quad x \in D_{i,j}(\zeta_m).$$

Lemma 2.6. In this proof, we put $p = p_i$, $q_j = q_{i,j}$, $b_1 = b_1^i$, $b_2 = b_2^i$, $\Theta_m = \Theta^{\mathrm{u}}_{i,j,m}$, $\theta_m = \theta^{\mathrm{u}}_{i,j,m}$ and $W_m = W^{\mathrm{u}}_{i,j,m}$ for short. From a simple calculation, we have the Jacobian matrix

$$\frac{Dx}{Dy} = \left({}^{\mathrm{t}}q_1 + \zeta_m \sum_{j=1}^{n} y_j \ {}^{\mathrm{t}}q_j{}', \zeta_m {}^{\mathrm{t}}q_2, \dots, \zeta_m {}^{\mathrm{t}}q_n \right),$$

$$\frac{Dx}{Dy}^{-1} = \begin{pmatrix} \dfrac{1}{1+\zeta_m\gamma_1} q_1 \\ -\dfrac{\gamma_2}{1+\zeta_m\gamma_1} q_1 + \dfrac{1}{\zeta_m} q_2 \\ \vdots \\ -\dfrac{\gamma_n}{1+\zeta_m\gamma_1} q_1 + \dfrac{1}{\zeta_m} q_n \end{pmatrix}$$

where $q_j = q_j(y_1)$, $q_j{}' = q_j{}'(y_1)$, $\gamma_k = \sum_{j=2}^{n} y_j \ q_j{}'(y_1) \cdot {}^{\mathrm{t}}q_k(y_1)$. From (2.10) and (2.15) we obtain

$$\begin{aligned} \Delta_x \Theta_m(x) + f(\Theta_m(x)) &= \frac{d^2\theta_m}{ds^2}(y_1) + \Delta_{\tilde{y}} W_m(y_1, \tilde{y}) + f(\Theta_m(x)) + O(\zeta_m) \\ &= -\zeta_m{}^{1/3} + \zeta_m{}^{2/3} \int_{\partial Q(y_1)} d\omega_{\tilde{\xi}} \\ &\quad + f(\Theta_m(x)) - f(\theta_m(y_1)) + O(\zeta_m) \\ &= -\zeta_m{}^{1/3} + O(\zeta_m{}^{2/3}) \quad \text{as } \zeta_m \to 0. \end{aligned}$$

Therefore, for small ζ_m we obtain

$$\Delta_x (\Theta_m - \Psi_m)(x) + h(x)(\Theta_m - \Psi_m)(x) \le 0 \quad \text{in } D_{i,j}(\zeta_m)$$

where $h(x) = \int_0^1 f'(t\Theta_m(x) + (1-t)\Psi_m(x))dt$.

Let $T = \partial D_{i,j}(\zeta_m) \setminus \overline{S_i(s_{i,j-1}, \zeta_m) \cup S_i(s_{i,j+1,\zeta_m})}$. From (2.13) and (2.15), we have

$$\begin{aligned} \nu(x) \cdot {}^{\mathrm{t}}\nabla_x (\Theta_m - \Psi_m) &= \nu(x) \cdot {}^{\mathrm{t}}\frac{Dx}{Dy}^{-1} \cdot {}^{\mathrm{t}}\left(\frac{d\theta_m}{ds}(y_1), 0, \dots, 0 \right) \\ &\quad + \zeta_m{}^2 \nu(x) \cdot {}^{\mathrm{t}}\frac{Dx}{Dy}^{-1} \cdot {}^{\mathrm{t}}\nabla_{(s,\tilde{y})} W_m(y_1, \tilde{y}) \\ &= \zeta_m{}^{1+2/3} a_i(s) + O(\zeta_m{}^2) \end{aligned}$$

on $x \in T$ as $\zeta_m \to 0$. In the case $1 \leq j \leq N(i) - 1$, for small ζ_m we have

$$\Theta_m(x) - \Psi_m(x) \geqq 0 \quad x \in \overline{S_i(s_{i,j-1}, \zeta_m) \cup S_i(s_{i,j+1,\zeta_m})}.$$

In the case $j = N(i)$, we have

$$\begin{aligned} &\Theta_m(x) - \Psi_m(x) \geqq 0 \quad x \in \overline{S_i(s_{i,j-1}, \zeta_m)}, \\ &\nu(x) \cdot {}^t\nabla_x(\Theta_m - \Psi_m) = \zeta_m + O({\zeta_m}^2) \quad x \in \overline{S_i(l_i, \zeta_m)} \quad \text{as } \zeta_m \to 0. \end{aligned}$$

Because of $|h(x)| \leq \sup_{|\xi|<M_1+1} |f'(\xi)|$ and (2.8), applying the strong maximum principle we obtain Lemma 2.6. □

From an argument similar to that of Lemma 2.5, we define $\theta^1_{i,j,m}$ as the unique solution of

$$\begin{cases} \dfrac{1}{a_i(s)} \dfrac{d}{ds}\left(a_i(s)\dfrac{d\psi}{ds}\right) + f(\psi) - {\zeta_m}^{1/3} = 0 \quad (s_{i,j-1} < s < s_{i,j+1}) \\ \psi(s_{i,j-1}) = \phi_{i,j-1} - \sup\limits_{x \in S_i(s_{i,j-1}, \zeta_m)} |\Psi_m(x) - \phi_{i,j-1}|, \\ \psi(s_{i,j+1}) = \phi_{i,j+1} - \sup\limits_{x \in S_i(s_{i,j+1}, \zeta_m)} |\Psi_m(x) - \phi_{i,j+1}| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in the case } 1 \leq j \leq N(i) - 1, \\ \dfrac{d\psi}{ds}(s_{i,j+1}) = -\zeta_m \qquad\qquad\qquad\qquad \text{in the case } j = N(i). \end{cases}$$

We define $W^1_{i,j,m}(s, \tilde{y})$ on $Q_i(s)$ ($s \in [s_{i,j-1}, s_{i,j+1}]$) by the solution of

$$\begin{cases} \Delta_{\tilde{y}} W = \dfrac{{a_i}'(s)}{a_i(s)} \theta^1_{i,j,m}(s) - {\zeta_m}^{2/3} \int_{\partial Q_i(s)} d\tilde{\xi}^{n-2} & \text{in } Q_i(s) \\ \dfrac{\partial W}{\partial \tilde{\nu}} = \dfrac{b^i_2(x)}{\zeta_m} \theta^1_{i,j,m}(s) - {\zeta_m}^{2/3} a_i(s) & \text{on } \partial Q_i(s) \end{cases}$$

satisfying $W(g(s, \tilde{y}^1)) = 1$ where $g(s, \cdot) : Q_i(s_{i,j}) \to Q_i(s)$ is C^3-diffeomorphism and we define $\Theta^1_{i,j,m}(x)$ ($x \in D_{i,j}(\zeta_m)$) by

$$\Theta^1_{i,j,m}(x) = \theta^1_{i,j,m}(y_1) + {\zeta_m}^2 W^1_{i,j,m}(y_1, \tilde{y}) - \zeta_m \quad x \in D_{i,j}(\zeta_m)$$

where $y = (y_1, \tilde{y})$ satisfies $x = p_i(y_1) + \zeta_m \sum_{j=1}^n y_j \; q_{i,j}(y_1)$. From an argument similar to the proof of Lemma 2.6, we have the following:

Lemma 2.7. *The function $\Theta^1_{i,j,m}(x)$ is a lower solution of Ψ_m restricted on $D_{i,j}(\zeta_m)$, that is,*

$$\Theta^1_{i,j,m}(x) \leq \Psi_m(x) \quad x \in D_{i,j}(\zeta_m).$$

We define $\theta_{i,j,\infty}(s)$ $(s_{i,j-1} \le s \le s_{i,j+1})$ by the limit of $\theta^{\mathrm{u}}_{i,j,m}$ as $m \to \infty$ where $s_{i,0} = 0$ for short. From the definition of $\theta^{\mathrm{u}}_{i,j,m}$ and $\theta^{\mathrm{l}}_{i,j,m}$, the function $\theta_{i,j,\infty}(s)$ satisfies

$$\begin{cases} \dfrac{1}{a_i(s)}\dfrac{d}{ds}\left(a_i(s)\dfrac{d\theta_{i,j,\infty}}{ds}\right) + f(\theta_{i,j,\infty}) = 0 & (s_{i,j-1} < s < s_{i,j+1}) \\ \theta_{i,j,\infty}(s_{i,j-1}) = \phi_{i,j-1}, & \\ \theta_{i,j,\infty}(s_{i,j+1}) = \phi_{i,j+1} & \text{in the case } 1 \le j \le N(i) - 1, \\ \dfrac{d\theta_{i,j,\infty}}{ds}(s_{i,j+1}) = 0 & \text{in the case } j = N(i). \end{cases}$$

and $\theta^{\mathrm{u}\,(1)}_{i,j,m}$ converge to $\theta_{i,j,\infty}$ uniformly on the interval $[s_{i,j-1}, s_{i,j+1}]$ as $m \to \infty$ where we put $\theta^{\mathrm{u}\,(1)}_{i,1,m}(s) = \theta^{\mathrm{u}\,(1)}_{i,1,m}(\zeta l)$ $(0 \le s \le \zeta l)$. Thus, Ψ_m restricted on $D_{i,j}(\zeta_m)$ satisfies

$$\sup_{x \in D_{i,j}(\zeta_m)} |\Psi_m(x) - \theta_{i,j,\infty}(s)| \to 0 \quad \text{as } m \to \infty$$

where s satisfies $S_i(s, \zeta_m) \ni x$. Moreover, we obtain $\theta_{i,j,\infty}(s) = \theta_{i,j+1,\infty}(s)$ $(s_{i,j} < s < s_{i,j+1})$ by Lemma 2.6 and Lemma 2.7. Indeed, we have

$$\begin{aligned} |\theta_{i,j,\infty}(s) - \theta_{i,j+1,\infty}(s)| &\le |\theta_{i,j,\infty}(s) - \Psi_m(x')| + |\Psi_m(x') - \theta_{i,j+1,\infty}(s)| \\ &\le \sup_{x \in D_{i,j}(\zeta_m)} |\Psi_m(x) - \theta_{i,j,\infty}(t)| + \sup_{x \in D_{i,j+1}(\zeta_m)} |\Psi_m(x) - \theta_{i,j+1,\infty}(t)| \\ &\to 0 \quad (m \to \infty) \end{aligned}$$

where $x' \in D_{i,j}(\zeta_m) \cap D_{i,j+1}(\zeta_m)$ satisfies $x' \in S_i(s, \zeta_m)$ and t satisfies $S_i(t, \zeta_m) \ni x$. We define $\psi_i(s)$ $(0 \le s \le l_i)$ by

$$\psi_i(s) = \theta_{i,j,\infty}(s) \quad (s_{i,j-1} \le s \le s_{i,j+1}), \quad 1 \le j \le N(i).$$

Then, $(\psi_1, \ldots, \psi_N)$ satisfies

$$\begin{cases} \dfrac{1}{a_i(s)}\dfrac{d}{ds}\left(a_i(s)\dfrac{d\psi_i}{ds}\right) + f(\psi_i) = 0 \quad (0 < s < l_i, \quad 1 \le i \le N), \\ \psi_1(0) = \cdots = \psi_N(0), \quad \dfrac{d\psi_i}{ds}(l_i) = 0 \quad (1 \le i \le N), \end{cases}$$

Ψ_m restricted on $D_i(\zeta_m)$ converges ψ_i uniformly and Ψ_m restricted on $J(\zeta_m)$ converges $\psi_i(0)$ uniformly as $m \to \infty$.

Lemma 2.8. *$(\psi_1, \ldots, \psi_N)$ satisfies*

$$\sum_{i=1}^{N} a_i(0)\frac{d\psi_i}{ds}(0) = 0.$$

Proof of Lemma 2.8. We have

$$\frac{1}{\zeta_m}^{n-1}\int_{\Omega(\zeta_m)} f(\Psi_m(x))dx = -\frac{1}{\zeta_m}^{n-1}\int_{\Omega(\zeta_m)} \Delta\Psi_m(x)dx$$
$$= -\frac{1}{\zeta_m}^{n-1}\int_{\partial\Omega(\zeta_m)} \frac{d\Psi_m}{d\nu}(x)$$
$$= 0.$$

Letting m tend to infinity, we obtain

$$\sum_{i=1}^{N}\int_0^{l_i} a_i(s)f(\psi_i(s))ds = 0.$$

Thus, we obtain

$$0 = -\sum_{i=1}^{N}\int_0^{l_i}\frac{d}{ds}\left\{a_i(s)\frac{\psi_i}{ds}(s)\right\}ds = \sum_{i=1}^{N} a_i(0)\frac{\psi_i}{ds}(0).$$

□

Therefore, we complete the proof of Theorem 2.1. □

§3. Inverse Problem

In this section, we consider a certain inverse problem. We have proved a solution of (1.1) approaches to a solution of an associated limit equation (2.5) as ζ tends to zero. In that situation, conversely, the following problem occurs naturally:

When a solution of (2.5) is given, can we prove the existence of a solution of (1.1) which approaches it?

We have a positive answer. We can prove that (1.1) on a simple network-shaped domain has a solution which approaches a solution of (2.5) when the solution of (2.5) satisfies a certain condition, that is, we have the following:

THEOREM 3.1. *Suppose that there exists a solution $\psi = (\psi_1, \ldots, \psi_n)$ of (2.5) such that the linearized equation*

$$(3.1)\qquad \begin{cases} \dfrac{1}{a_i(s)}\dfrac{d}{ds}\left(a_i(s)\dfrac{d\phi_i}{ds}\right) + f'(\psi_i(s))\phi_i = 0 \quad (0 < s < l_i), \quad 1 \leqq i \leqq N, \\ \phi_1(0) = \cdots = \phi_n(0), \quad \sum\limits_{i=1}^{N} a_i(0){\phi_i}'(0) = 0, \\ \dfrac{d\phi_i}{ds}(l_i) = 0, \quad 1 \leqq i \leqq N \end{cases}$$

has no solution except the trivial solution $(\phi_1, \dots, \phi_n) = (0, \dots, 0)$*, namely, we suppose the eigenvalue problem of the linearized equation around* ψ *has no zero eigenvalue. Then, there exists a constant* $\zeta_* > 0$ *such that the equation* (1.1) *has a solution* Ψ_ζ *for any* $\zeta \in (0, \zeta_*]$ *and that* $\{\Psi_\zeta : 0 < \zeta < \zeta_*\}$ *satisfies*

$$\begin{cases} \lim_{\zeta \to 0} \sup_{x \in J(\zeta)} |\Psi_\zeta(x) - \psi_i(0)| = 0 & \text{for } 1 \leq i \leq N, \\ \lim_{\zeta \to 0} \sup_{x \in D_i(\zeta)} |\Psi_\zeta(x) - \psi_i(s)| = 0 & \text{for } 1 \leq i \leq N \end{cases}$$

where $s \in (l\zeta, l_i)$ *defined by* $S_i(s, \zeta) \ni x$*.*

PROOF OF THEOREM 3.1. We construct an approximate solution of (1.1). Let a solution $\psi = (\psi_1, \dots, \psi_n)$ of (2.5) satisfy the assumption of Theorem 3.1. We define a Lipschitz continuous function $\Psi_\zeta^{(0)}$ as

$$\Psi_\zeta^{(0)}(x) = \begin{cases} \psi_1(0) & x \in J(\zeta), \\ \psi_i(l_i(s - \zeta l)/(l_i - \zeta l)) & x \in D_i(\zeta) \quad \text{for } 1 \leq i \leq N \end{cases}$$

where $s \in (l\zeta, l_i)$ satisfies $S_i(s, \zeta) \ni x$.

After this, let $\|\cdot\|_\zeta$ denote a norm $\|g\|_\zeta = \sup_{x \in \Omega(\zeta)} |g(x)|$ of $C^0(\overline{\Omega(\zeta)})$.

LEMMA 3.2. *There exists a constant* $\zeta' > 0$ *such that if* Φ_ζ *satisfies*

$$\begin{cases} \Delta \Phi_\zeta + f'(\Psi_\zeta^{(0)}(x))\Phi_\zeta = 0 & \text{in } \Omega(\zeta), \\ \dfrac{\partial \Phi_\zeta}{\partial \nu} = 0 & \text{on } \partial\Omega(\zeta) \end{cases} \tag{3.2}$$

for any $\zeta \in (0, \zeta']$*, then* $\Phi_\zeta \equiv 0$ *in* $\Omega(\zeta)$*.*

PROOF OF LEMMA 3.2. Suppose that there exists a positive sequence $\{\zeta_m\}_{m=1}^\infty$ with $\lim_{m \to \infty} \zeta_m = 0$ such that the equation (3.2) at $\zeta = \zeta_m$ has a nontrivial solution $W_m \not\equiv 0$ in $\Omega(\zeta_m)$. Let $\widetilde{W}_m(x) = W_m(x)/\|W_m\|_{\zeta_m}$. Clearly, $\widetilde{W}_m$ satisfies (3.2) and $\|\widetilde{W}_m\|_{\zeta_m} = 1$ for any $m \geqq 1$.

From an argument similar to the proof of Theorem 2.1, we obtain a nontrivial solution of (3.1). This contradicts the assumption of Theorem 3.1.

Thus we complete the proof of Lemma 3.2. □

For $\Phi_\zeta \in L^2(\Omega(\zeta))$ we consider the equation

$$\begin{cases} \Delta u + f'(\Psi_\zeta^{(0)})u = \Phi_\zeta & \text{in } \Omega(\zeta), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega(\zeta). \end{cases} \tag{3.3}$$

From Lemma 3.2, the equation (3.3) has a unique solution for each Φ_ζ. We denote by $A_\zeta \Phi_\zeta$ the solution of (3.3) for Φ_ζ.

LEMMA 3.3. *There exist constants $M_6 > 0$ and $\zeta'' > 0$ such that*

$$\|A_\zeta \Phi_\zeta\|_\zeta \leqq M_6 \|\Phi_\zeta\|_\zeta$$

for any $\zeta \in (0, \zeta'']$ and $\Phi_\zeta \in C^0(\overline{\Omega(\zeta)})$ satisfying $A_\zeta \Phi \in C^2(\Omega(\zeta)) \cap C^0(\overline{\Omega(\zeta)})$.

PROOF OF LEMMA 3.3. We assume the contrary, that is, assume there exist a sequence $\{\zeta_m\}_{m=1}^{\infty}$ with $\lim_{m\to\infty} \zeta_m = 0$ and C^0 functions Θ_m such that $\|\Theta_m\|_{\zeta_m} = 1$ and $\|A_{\zeta_m}\Theta_m\|_{\zeta_m} \to \infty$ for $m \to \infty$. Let

$$U_m(x) = \frac{A_{\zeta_m}\Theta_m(x)}{\|A_{\zeta_m}\Theta_m\|_{\zeta_m}}, \quad \widetilde{\Theta}_m(x) = \frac{\Theta_m(x)}{\|A_{\zeta_m}\Theta_m\|_{\zeta_m}}.$$

Then, $(U_m, \widetilde{\Theta}_m)$ satisfies

$$\begin{cases} \Delta U_m + f'(\Psi_{\zeta_m}^{(0)})U_m = \widetilde{\Theta}_m & \text{in } \Omega(\zeta_m), \\ \dfrac{\partial U_m}{\partial \nu} = 0 & \text{on } \partial\Omega(\zeta_m), \end{cases}$$
$$\|U_m\|_{\zeta_m} = 1, \quad \|\widetilde{\Theta}_m\|_{\zeta_m} \to 0 \quad \text{as } m \to \infty.$$

From an argument similar to the proof of Theorem 2.1, we obtain a nontrivial solution of (3.1). This contradicts the assumption of Theorem 3.1. Thus we complete the proof of Lemma 3.3. □

We define a sequence $\{\Psi_\zeta^{(p)}\}_{p=0}^{\infty} \subset C^0(\overline{\Omega(\zeta)})$ as

$$\Psi_\zeta^{(p+1)} = A_\zeta \left(f'(\Psi_\zeta^{(0)})\Psi_\zeta^{(p)} - f(\Psi_\zeta^{(p)}) \right) \quad \text{for } p \geqq 0.$$

From Schauder estimates and Theorem 4.45 of Troianiello [11], we remark $\Psi_\zeta^{(p)} \in C^2(\Omega(\zeta)) \cap C^0(\overline{\Omega(\zeta)})$.

We take a constant $\delta > 0$ such that

$$\delta < \min\left\{1, \left(2M_6 \sup_{|\xi|<M_1+2} |f''(\xi)|\right)^{-1}\right\}. \tag{3.4}$$

Then, we have the following:

LEMMA 3.4. *There exists a positive constant ζ_* such that*

$$\left\|\Psi_\zeta^{(p)} - \Psi_\zeta^{(0)}\right\|_\zeta \leq \delta \tag{3.5}$$

for any $p \geqq 1$ and $\zeta \in (0, \zeta_*]$.

PROOF OF LEMMA 3.4. We prove Lemma 3.4 by the induction. From $\|\Psi_\zeta^{(1)}\|_\zeta \le M_6\|f'(\Psi_\zeta^{(0)})\Psi_\zeta^{(0)} - f(\Psi_\zeta^{(0)})\|_\zeta \le M_6\left(\sup_{|\xi|<M_1}|f'(\xi)|M_1 + \sup_{|\xi|<M_1}|f(\xi)|\right)$, there exists a solution $\psi^{(1)} = (\psi_1^{(1)}, \cdots, \psi_N^{(1)})$ of

$$\begin{cases} \dfrac{1}{a_i(s)}\dfrac{d}{ds}\left(a_i(s)\dfrac{d\psi_i^{(1)}}{ds}\right) + f'(\psi_i(s))\psi_i^{(1)} = f'(\psi_i(s))\psi_i(s) - f(\psi_i(s)) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{on } 0 < s < l_i \quad \text{for } 1 \leqq i \leqq N, \\ \psi_1^{(1)}(0) = \cdots = \psi_N^{(1)}(0), \quad \displaystyle\sum_{i=1}^N a_i(0)\frac{d\psi_i^{(1)}}{ds}(0) = 0, \\ \dfrac{d\psi_i^{(1)}}{ds}(l_i) = 0 \quad \text{for } 1 \leqq i \leqq N \end{cases}$$

and $\Psi_\zeta^{(1)}$ converges to $\psi^{(1)}$ as $\zeta \to 0$ by an argument similar to the proof of Theorem 2.1. Thus, $\psi - \psi^{(1)} = (\psi_1 - \psi_1^{(1)}, \ldots, \psi_N - \psi_N^{(1)})$ satisfies (3.1). Therefore we obtain $\psi = \psi^{(1)}$ and $\|\Psi_\zeta^{(1)} - \Psi_\zeta^{(0)}\|_\zeta \to 0$ as $\zeta \to 0$.

Let $\zeta_* > 0$ be a small constant satisfying

$$\|\Psi_\zeta^{(1)} - \Psi_\zeta^{(0)}\|_\zeta \le \delta/2 \quad \text{for } \zeta \in (0, \zeta_*].$$

We assume $\Psi_\zeta^{(p)}$ satisfies (3.5). Then, we have

$$\|\Psi_\zeta^{(p+1)} - \Psi_\zeta^{(0)}\|_\zeta \leqq \|\Psi_\zeta^{(p+1)} - \Psi_\zeta^{(1)}\|_\zeta + \|\Psi_\zeta^{(1)} - \Psi_\zeta^{(0)}\|_\zeta$$

and from (3.4) and (3.5) we have

$$\begin{aligned} &\|\Psi_\zeta^{(p+1)} - \Psi_\zeta^{(1)}\|_\zeta \\ &= \left\|A_\zeta\left(f'(\Psi_\zeta^{(0)})(\Psi_\zeta^{(p)} - \Psi_\zeta^{(0)}) - \left(f(\Psi_\zeta^{(p)}) - f(\Psi_\zeta^{(0)})\right)\right)\right\|_\zeta \\ &\le M_6\left\|\int_0^1 \left\{f'(\Psi_\zeta^{(0)}) - f'(t\Psi_\zeta^{(p)} + (1-t)\Psi_\zeta^{(0)})\right\} dt(\Psi_\zeta^{(p)} - \Psi_\zeta^{(0)})\right\|_\zeta \\ &\le M_6\left\|\int_0^1\int_0^1 f''\left(\Psi_\zeta^{(0)} + t(1-t_1)(\Psi_\zeta^{(p)} - \Psi_\zeta^{(0)})\right) t dt_1 dt(\Psi_\zeta^{(p)} - \Psi_\zeta^{(0)})^2\right\|_\zeta \\ &\le M_6 \sup_{|\xi|<M_1+2}|f''(\xi)|\delta^2 \le \delta/2 \end{aligned}$$

So, we have $\left\|\Psi_\zeta^{(p+1)} - \Psi_\zeta^{(0)}\right\|_\zeta \le \delta$ for $\zeta \in (0, \zeta_*]$. We complete the proof of Lemma 3.4. □

From Lemma 3.4, we have $\|\Psi_\zeta^{(p+1)} - \Psi_\zeta^{(p)}\|_\zeta \leqq 2^{-1}\|\Psi_\zeta^{(p)} - \Psi_\zeta^{(p-1)}\|_\zeta \leqq \delta 2^{-p}$ for any $p \geqq 1$. We have immediately that the sequence $\{\Psi_\zeta^{(p)}\}_{p=1}^\infty$ is a Cauchy sequence in $C^0(\overline{\Omega(\zeta)})$. We denote by Ψ_ζ the limit of $\Psi_\zeta^{(p)}$ as $p \to \infty$. We obtain

$\Psi_\zeta = A_\zeta(f'(\Psi_\zeta^{(0)})\Psi_\zeta - f(\Psi_\zeta)) \in C^2(\Omega(\zeta))$. So, Ψ_ζ satisfies (1.1). On the other hand, we obtain

$$\begin{aligned}\|\Psi_\zeta - \Psi_\zeta^{(0)}\|_\zeta &\leqq \|\Psi_\zeta - \Psi_\zeta^{(1)}\|_\zeta + \|\Psi_\zeta^{(1)} - \Psi_\zeta^{(0)}\|_\zeta \\ &\leqq 2^{-1}\|\Psi_\zeta - \Psi_\zeta^{(0)}\|_\zeta + \|\Psi_\zeta^{(1)} - \Psi_\zeta^{(0)}\|_\zeta.\end{aligned}$$

Therefore, $\|\Psi_\zeta - \Psi_\zeta^{(0)}\|_\zeta \leqq 2\|\Psi_\zeta^{(1)} - \Psi_\zeta^{(0)}\|_\zeta \to 0$ as $\zeta \to 0$. We complete the proof of Theorem 3.1. □

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